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Weakly Convergent Sequences of Functions and Orthogonal Polynomials*

PAUL NEVAI[†]

Department of Mathematics, The Ohio State University, P.O. Box 3341, Columbus, Ohio 43210–0341, U.S.A.

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Measures and sequences of functions on locally compact spaces are studied, and a condition is given that, for a sequence of functions that is weakly convergent in L^1, ensures the strong convergence of a related sequence of functions. This result, together with a new integral formula for the reflection coefficients $\Phi_n(0)$ for the monic orthogonal polynomial Φ_n associated with a measure on the unit circle, is used to investigate convergence properties of orthogonal polynomials.

CONTENTS

1. INTRODUCTION

In this paper we study measures and sequences of functions on locally compact spaces, and consequences of these results for, and other related results involving, orthogonal polynomials. The result we obtain for sequences of functions can be described as Tauberian in spirit, since in its main

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[†] E-mail address: nevai@mps.ohio-state.edu or nevai@ohstpy.bitnet

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application it enables one to conclude the strong convergence of a sequence of functions given that this sequence is weakly convergent. This result relies on a lemma explaining how to deal with the part of a measure that is singular with respect to another measure. Sections 2 and 3 presenting these matters are readable without any knowledge of orthogonal polynomials.

As for the part of the paper discussing orthogonal polynomials, consider a finite positive Borel measure μ on the interval $[0, 2\pi)$ such that its support, $\operatorname{supp}(\mu)$, is an infinite set. The polynomials $\phi_n(z) = \phi_n(d\mu, z) = \kappa_n(d\mu) z^n + \cdots$ orthonormal with respect to μ on the unit circle are defined by the requirements that $\kappa_n = \kappa_n(d\mu) > 0$ and

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_m(d\mu, e^{it}) \overline{\phi_n(d\mu, e^{it})}\, d\mu(t) = \delta_{mn}. \tag{1}$$

The monic orthogonal polynomials $\Phi_n = \Phi_n(d\mu)$ are defined by $\Phi_n = \kappa_n^{-1} \phi_n$. The *Erdös class* of measures are measures μ for which $\mu'(t) > 0$ for almost every $t \in [0, 2\pi)$. An important result of E. A. Rahmanov [7, Theorem, p. 106] for measures in the Erdös class is the following

THEOREM 1. *Assume $\mu'(t) > 0$ for almost every $t \in [0, 2\pi)$. Then*

$$\lim_{n \to \infty} \Phi_n(d\mu, 0) = 0. \tag{2}$$

A simplified proof of this result was given in [4] and subsequently in [8]; the purpose of this paper is to further simplify the proof. Before we can discuss this new proof, we need the concept of the *-operator. If p is a polynomial of degree n, then p^* is defined by

$$p^*(z) = z^n \overline{p(1/\bar{z})}. \tag{3}$$

The *-operator can be used to define the reverse polynomials $\phi_n^*(d\mu)$ and $\Phi_n^*(d\mu)$.

In what follows, H^1 will, as usual, denote the Hardy space of functions f holomorphic in the open unit disk $\{z : |z| < 1\}$ such that $\sup_{r<1} \int_0^{2\pi} |f(re^{it})|\, dt < \infty$.

The key ingredient of our new proof is the following result, which is likely to have other applications in the theory of orthogonal polynomials on the unit circle.

THEOREM 2. *Let μ be an arbitrary finite positive Borel measure on the* *interval* $[0, 2\pi)$ *such that* $\operatorname{supp}(\mu)$ *is an infinite set. Then for every* $f \in H^1$ *we have*

$$-\Phi_{n+1}(d\mu, 0) = \frac{1}{2\pi} \int_0^{2\pi} z \frac{\phi_n(d\mu, z)}{\phi_n^*(d\mu, z)} \times \left(\frac{|\phi_n(d\mu, z)|^2}{|\phi_{n+l}(d\mu, z)|^2} - f(z) \right) dt \qquad (z = e^{it}) \tag{4}$$

for all integers $n \geq 0$ *and* $l > 0$.

Using this theorem, we will be able to give precise estimates for $\Phi_{n+1}(d\mu, 0)$:

THEOREM 3. *Let* μ *be a finite positive Borel measure in the interval* $[0, 2\pi)$ *such that* $\operatorname{supp}(\mu)$ *is an infinite set. Then, writing* $z = e^{it}$, *we have*

$$\begin{aligned} |\Phi_{n+1}(d\mu, 0)| &\leq \inf_{\substack{l > 1 \\ f \in H^1}} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|\phi_n(d\mu, z)|^2}{|\phi_{n+l}(d\mu, z)|^2} - f(z) \right| dt \\ &\leq \inf_{c \text{ real}} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{|\phi_n(d\mu, z)|^2}{|\phi_{n+1}(d\mu, z)|^2} - c \right| dt \\ &\leq |\Phi_{n+1}(d\mu, 0)| \sqrt{\frac{2}{1 + |\Phi_{n+1}(d\mu, 0)|^2}} \\ &\leq \sqrt{2}\, |\Phi_{n+1}(d\mu, 0)| \end{aligned} \tag{5}$$

for $n = 0, 1, \ldots$.

Remark. If μ is the Lebesgue measure then the first inequality in (5) turns into equality for all $n = 0, 1, \ldots$. More generally, if $d\mu = g\, dt$, where g is the reciprocal of a positive trigonometric polynomial of degree, say, m, then the first inequality in (5) turns into equality for all $n = m, m+1, \ldots$ (cf. [1, Theorem 5.4.5, p. 224] or [10, Theorem 11.2, p. 289]). In addition, since $|\Phi_{n+1}(d\mu, 0)|$ can be made arbitrarily close to 1, the third inequality in (5) cannot be improved. Finally, since $|\Phi_{n+1}(d\mu, 0)|$ can be made arbitrarily close to 0 as well, the last inequality in (5) cannot be improved either.

The following result summarizes the subtle difference between the conditions $\mu' > 0$ a.e. in $[0, 2\pi)$ and $\lim_{n \to \infty} \Phi_n(d\mu, 0) = 0$, and gives a complete characterization of these conditions in terms of L_1 ratio asymptotics of the corresponding orthogonal polynomials.

THEOREM 4. *Let μ be a finite positive Borel measure in the interval* $[0, 2\pi)$ *such that* $\operatorname{supp}(\mu)$ *is an infinite set. Then*

$$\lim_{n\to\infty} \Phi_n(d\mu, 0)=0$$
$$\Leftrightarrow \lim_{n\to\infty} \inf_{l\geqslant 1} \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{|\phi_n(d\mu, e^{it})|^2}{|\phi_{n+l}(d\mu, e^{it})|^2} - 1 \right| dt = 0 \tag{6}$$

and

$$\mu'(t)>0 \text{ a.e. in } [0, 2\pi)$$
$$\Leftrightarrow \lim_{n\to\infty} \sup_{l\geqslant 1} \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{|\phi_n(d\mu, e^{it})|^2}{|\phi_{n+l}(d\mu, e^{it})|^2} - 1 \right| dt = 0. \tag{7}$$

In what follows we will not always indicate explicitly the dependence of the polynomials ϕ_n, Φ_n and so forth on μ; that is, we will write, for instance, $\phi_n(z)$ instead of $\phi_n(d\mu, z)$. We will follow this practice as long as there is no danger that it will lead to confusion.

2. ANNIHILATING THE SINGULAR PARTS OF MEASURES

Our main result about locally compact spaces will consider two measures ρ and σ and a sequence of functions g_n such that $g_n^2\, d\rho$ weakly converges to $d\sigma$, as described below. In the proof of this result the part of σ that is singular with respect to ρ plays a special role. The aim of this section is to show how to deal with this singular part. To this end, let S be a locally compact Hausdorff topological space. We recall that a positive Borel measure ρ on S is a positive measure on the Borel sets of S such that $\rho(C)<\infty$ for every compact set C. ρ is called regular if

$$\rho(X)=\inf\{\rho(O)\colon X\subset O,\ O \text{ is open}\} \tag{8}$$

and

$$\rho(X)=\sup\{\rho(C)\colon C\subset X,\ C \text{ is compact}\} \tag{9}$$

hold for every Borel set $X\subset S$. We have the following result, which is a generalization of Lemma 1 of [4, p. 65].

LEMMA 5. *Let S be a locally compact Hausdorff space, and let ρ be a positive regular Borel measure on S. Let σ be a finite positive regular Borel measure on S that is singular with respect to ρ. Then there is a sequence*

$\langle h_n \rangle$ of real-valued functions continuous on S such that $0 \leqslant h_n(x) \leqslant 1$ for all $x \in S$ and for all integers $n \geqslant 0$,

$$\lim_{n \to \infty} h_n(x) = 1 \qquad \text{a.e. with respect to } \rho \tag{10}$$

and

$$\lim_{n \to \infty} \int h_n(x)\, d\sigma(x) = 0. \tag{11}$$

Proof. Let E be a Borel set such that $\rho(E) = 0$ and $\sigma(E) = \sigma(S)$; let $\langle E_n \rangle$ be decreasing sequence of open sets such that $\rho(E_n) < 1/n$ and $E \subset E_n$. For every positive integer n, let C_n be a compact set such that $C_n \subset E$ and $\sigma(C_n) > \sigma(E) - 1/n = \sigma(S) - 1/n$; such a C_n exists in view of the regularity of σ.

Fix $n \geqslant 1$. Let h_n be a continuous function mapping S into the interval $[0, 1]$ such that $h_n(x) = 0$ for $x \in C_n$ and $h_n(x) = 1$ for $x \in S \backslash E_n$; as for the existence of such an h_n, see, e.g., [3, Theorem 18 in Sect. 5, p. 146]. Then $\lim_{n \to \infty} h_n(x) = 1$ holds for every $x \in S \backslash \bigcap_{n=1}^{\infty} E_n$, i.e., almost everywhere with respect to ρ, showing that (10) holds. Moreover, we have

$$\int h_n(x)\, d\sigma(x) \leqslant \sigma(S \backslash C_n) < 1/n;$$

thus (11) also holds. The proof is complete. ∎

3. A Tauberian Result Concerning Weak Convergence

Given two measures on the same measurable space, denote by $d\sigma/d\rho$ the Radon–Nikodym derivative with respect to $d\rho$ of the part of $d\sigma$ that is absolutely continuous with respect to $d\rho$ (cf. [2, Sect. 32, pp. 132–135]). The following result has an application to orthogonal polynomials. This result and its proof below were motivated by results obtained in the special case concerning orthogonal polynomials on the unit circle (cf. [5, formula (4.5), p. 249] and [4, Theorem 3, p. 64]).

THEOREM 6. *Let S be a locally compact Hausdorff space, and let ρ and σ be positive regular Borel measures on S; assume that σ is finite. Put*

$$\Omega = \left\{ x \in S : \frac{d\sigma}{d\rho}(x) > 0 \right\}$$

and assume that $\rho(\Omega) < \infty$. *Let* $\langle g_n \rangle$ *be a sequence of nonnegative real-valued Borel measurable functions on* S *such that*

$$\limsup_{n \to \infty} \int f(x)\, g_n^2(x)\, d\rho(x) \leqslant \int f(x)\, d\sigma(x) \tag{12}$$

holds for every nonnegative function f *continuous on* S *and having compact support. Then*

$$\liminf_{n \to \infty} \int \frac{1}{g_n(x)} \sqrt{\frac{d\sigma}{d\rho}(x)}\, d\rho(x) \geqslant \rho(\Omega); \tag{13}$$

in this formula, one should take $1/0 = +\infty$ *and* $\infty \cdot 0 = 0$.

The assumption that the measures ρ and σ are regular is a natural one. In fact, Eq. (12) describes σ through specifying the integral of continuous functions with respect to σ; this, however, does not determine σ uniquely but for the assumption of regularity, since integrals of continuous functions determine σ only on Baire sets (cf. [2, Theorem D, Sect. 54, p. 239]; see also the Riesz Representation Theorem in [9, Theorem 2.14, p. 40]).

Proof. We may assume that $\rho(\Omega) > 0$. Let f be an arbitrary real-valued continuous function on S having compact support. By Hölder's inequality, we have

$$\left(\int_\Omega \left(f(x) \frac{d\sigma}{d\rho}(x)\right)^{1/4} d\rho(x)\right)^4$$
$$\leqslant \left(\int_\Omega \frac{1}{g_n(x)} \sqrt{\frac{d\sigma}{d\rho}(x)}\, d\rho(x)\right)^2 \cdot \rho(\Omega) \cdot \int f(x)\, g_n^2(x)\, d\rho(x).$$

In the last integral we extended the domain of integration from Ω to S; this is clearly legitimate, since this may only increase the right-hand side. Making $n \to \infty$, we obtain by (12) that

$$\left(\int \left(f(x) \frac{d\sigma}{d\rho}(x)\right)^{1/4} d\rho(x)\right)^4$$
$$\leqslant \left(\liminf_{n \to \infty} \int \frac{1}{g_n(x)} \sqrt{\frac{d\sigma}{d\rho}(x)}\, d\rho(x)\right)^2 \cdot \rho(\Omega) \cdot \int f(x)\, d\sigma(x). \tag{14}$$

Here we extended the domains of all integrals from Ω to S; this did not make any difference, since the integrands outside Ω were zero in the affected integrals. Now write σ_s for the part of σ that is singular with respect to ρ. Then, clearly,

$$\int f(x)\, d\sigma(x) = \int f(x) \frac{d\sigma}{d\rho}(x)\, d\rho(x) + \int f(x)\, d\sigma_s(x). \tag{15}$$

Let now $\langle h_n \rangle$ be a sequence of functions satisfying the conclusion of Lemma 5 with σ_s replacing σ. Substituting $h_n f_n$ for f in (14) and then making $n \to \infty$, and using (15) and Lebesgue's Dominated Convergence Theorem, we obtain that

$$\left(\int \left(f(x)\frac{d\sigma}{d\rho}(x)\right)^{1/4} d\rho(x)\right)^4$$
$$\leqslant \left(\liminf_{n\to\infty} \int \frac{1}{g_n(x)} \sqrt{\frac{d\sigma}{d\rho}(x)}\, d\rho(x)\right)^2$$
$$\cdot \rho(\Omega) \cdot \int f(x) \frac{d\sigma}{d\rho}(x)\, d\rho(x). \tag{16}$$

Let $\varepsilon > 0$. By Lusin's theorem (see, e.g., [9, Theorem 2.24, p. 55]), there is a sequence of nonnegative functions $\{f_k\}$ continuous on S and having compact supports such that $f_k(x) \leqslant 1/\varepsilon$ for every $x \in S$ and

$$\lim_{k\to\infty} f_k(x) = \left(\frac{d\sigma}{d\rho}(x) + \varepsilon\right)^{-1} \qquad \text{in measure on } \Omega.$$

Substituting f_k for f in (16), using the version of Lebesgue's Dominated Convergence Theorem for convergence in measure rather than a.e. convergence, making $\varepsilon \to 0$, and using Lebesgue's theorem again (now for a.e. convergence), we obtain

$$(\rho(\Omega))^4 \leqslant \left(\liminf_{n\to\infty} \int \frac{1}{g_n(x)} \sqrt{\frac{d\sigma}{d\rho}(x)}\, d\rho(x)\right)^2 \cdot (\rho(\Omega))^2.$$

Therefore (13) follows. The proof is complete. ∎

4. Known Consequences for Orthogonal Polynomials

As we mentioned above, Theorem 6 has important consequences for orthogonal polynomials. We are going to discuss these consequences in this section. The key result is the following

LEMMA 7. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$, and assume that $\mu' > 0$ almost everywhere in this interval. Then*

$$\liminf_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |\phi_n(d\mu, z)| \sqrt{\mu'(t)}\, dt \geqslant 1 \qquad (z = e^{it}) \tag{17}$$

and

$$\liminf_{n\to\infty}\ \inf_{l\geqslant -n} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\phi_n(d\mu, z)|}{|\phi_{n+l}(d\mu, z)|}\,dt \geqslant 1 \qquad (z=e^{it}). \tag{18}$$

Proof. According to [1, Theorem V.2.2, p. 198] and its complex conjugate, we have

$$\int_0^{2\pi} \frac{z^k}{|\phi_n(z)|^2}\,dt = \int_0^{2\pi} z^k\,d\mu(t) \qquad (z=e^{it}) \tag{19}$$

for all integers k and n with $|k| \leqslant n$. Hence for every 2π-periodic continuous function f we have

$$\lim_{n\to\infty}\int_0^{2\pi} f(t)\frac{1}{|\phi_n(z)|^2}\,dt = \int_0^{2\pi} f(t)\,d\mu(t) \qquad (z=e^{it}); \tag{20}$$

indeed, this follows from (19) by the Weierstrass Approximation Theorem (use the case $k=0$ of (19) to estimate the integral involving the error of approximation). Now we use Theorem 6; for the space S in Theorem 6 take the real line modulo 2π, and use this result with $1/|\phi_n(e^{it})|$, the Lebesgue measure on $[0, 2\pi)$, and μ (defined in a natural way on S) replacing $g_n(t)$, ρ, and σ, respectively (note that on the real line every Borel measure is regular, since every Borel set on the real line is also a Baire set; cf. [2, Theorem E, Sect. 50, p. 218, and Theorem G, Sect. 52, p. 228]). With this substitution, (12) holds according to (20). Therefore (13) also holds, and this implies (17).

We are now going to show that (17) implies (18). Indeed, writing $z=e^{it}$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |\phi_n(z)|\sqrt{\mu'(t)}\,dt\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi} \frac{|\phi_n(z)|^{1/2}}{|\phi_{n+l}(z)|^{1/2}}\cdot|\phi_n(z)|^{1/2}(\mu'(t))^{1/4}\cdot|\phi_{n+l}(z)|^{1/2}(\mu'(t))^{1/4}\,dt\\
&\quad\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}\,dt\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_n(z)|^2\,\mu'(t)\,dt\right)^{1/4}\\
&\qquad\times\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_{n+l}(z)|^2\,\mu'(t)\,dt\right)^{1/4}
\end{aligned}\tag{21}$$

according to Hölder's inequality. Moreover,

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)|^2\,\mu'(t)\,dt \leqslant \frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)|^2\,d\mu(t)=1 \qquad (z=e^{it}) \tag{22}$$

according to the orthogonality relations (1). In virtue of this with $m=n$ and $m=n+l$, (21) entails

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}\,dt \geqslant \left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_n(z)|\sqrt{\mu'(t)}\,dt\right)^2 \qquad (z=e^{it}). \tag{23}$$

Therefore (18) follows from (17). The proof is complete. ∎

Following closely the argument given in [4, p. 68], we can use (18) to derive the following, originally given in [4, Theorem 3, p. 64].

THEOREM 8. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$ such that $\mu'>0$ almost everywhere in this interval. Then*

$$\lim_{n\to\infty}\sup_{l\geqslant 1}\int_0^{2\pi}\left|\frac{|\phi_n(d\mu, z)|^2}{|\phi_{n+l}(d\mu, z)|^2}-1\right|dt=0 \qquad (z=e^{it}). \tag{24}$$

Proof. For all integers $l\geqslant 1$ and $n\geqslant 0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}\,dt=\frac{1}{2\pi}\int_0^{2\pi}|\phi_n(z)|^2\,d\mu(t)=1 \qquad (z=e^{it}) \tag{25}$$

according to (19) and (1). Using this together with (18), the relation

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-1\right)^2 dt=0 \qquad (z=e^{it}) \tag{26}$$

can be obtained by multiplying out the square. By Schwarz's inequality, we have

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}-1\right|dt\right)^2$$
$$\leqslant\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}+1\right)^2 dt\cdot\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-1\right)^2 dt \qquad (z=e^{it}).$$

The first factor on the right-hand side here is less than 4 by (25) and the inequality $(a+b)^2\leqslant 2a^2+2b^2$. Hence (24) follows in virtue of (26). The proof is complete. ∎

Remark. Formula (24) is equivalent to the condition that $\mu'>0$ almost everywhere in $[0, 2\pi)$ (cf. Theorem 4 and [6, Theorem 1.1, p. 295]).

Another consequence of Lemma 7 is the following

THEOREM 9. *Let μ be a finite positive Borel measure on the interval $[0, 2\pi)$, and assume that $\mu' > 0$ almost everywhere in this interval. Then*

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} (|\phi_n(d\mu, z)| \sqrt{\mu'(t)} - 1)^2 \, dt = 0 \qquad (z = e^{it}) \tag{27}$$

and

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \left| |\phi_n(d\mu, z)|^2 \mu'(t) - 1 \right| dt = 0 \qquad (z = e^{it}). \tag{28}$$

These results were given in [5, Theorem 2.1 and Corollary 2.2., pp. 242–243].

Proof. Formula (27) can be obtained by multiplying out the square and then using (18) and (22), whereas formula (28) follows from (17), (22), and (27) in exatly the same way as we derived (24) from (18), (25), and (26) just before. The proof is complete. ∎

5. PROOF OF THEOREM 2

For the proof of Theorem 2 we need the following

LEMMA 10. *Let μ be an arbitrary finite positive Borel measure on the interval* $[0, 2\pi)$ *such that* $\operatorname{supp}(\mu)$ *is an infinite set. Then we have*

$$-\Phi_{n+1}(d\mu, 0) = \frac{1}{2\pi} \int_0^{2\pi} \phi_n^2(d\mu, z)\, z^{1-n} \, d\mu(t) \qquad (z = e^{it}) \tag{29}$$

for $n = 0, 1, \ldots$.

Proof. Writing $z = e^{it}$, we have

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \phi_n^2(z)\, z^{1-n} \, d\mu(t) \\ &\quad = \frac{1}{2\pi} \int_0^{2\pi} \phi_n(z)\, z^{1-n} \phi_n(z) \, d\mu(t) \\ &\quad = \frac{1}{2\pi} \int_0^{2\pi} \phi_n(z) \left(\kappa_n z + \sum_{j=0}^{n-1} \eta_{n-j-1,n} z^{-j} \right) d\mu(t), \end{aligned} \tag{30}$$

where to get the last equation we used the notation

$$\phi_n(z) = \kappa_n z^n + \sum_{l=0}^{n-1} \eta_{ln} z^l. \tag{31}$$

Now we have $\int_0^{2\pi} \phi_n(z)\, z^{-j}\, d\mu = 0$ for $0 \leqslant j < n$ in view of the orthogonality relations (1); hence the right-hand side of (30) equals

$$\frac{1}{2\pi} \int_0^{2\pi} \kappa_n z \phi_n(z)\, d\mu(t).$$

We can calculate this integral by using the recurrence formula

$$\kappa_n \zeta \phi_n(\zeta) = \kappa_{n+1} \phi_{n+1}(\zeta) - \phi_{n+1}(0)\, \phi_{n+1}^*(\zeta) \tag{32}$$

(cf. [1, formula (V.1.20), p. 195] or [10, formula (11.4.6), p. 293]). According to this, the expression in (30) equals

$$\frac{\kappa_{n+1}}{2\pi} \int_0^{2\pi} \phi_{n+1}(z)\, d\mu(t) - \frac{\phi_{n+1}(0)}{2\pi} \int_0^{2\pi} \phi_{n+1}^*(z)\, d\mu(t).$$

The first integral here is zero according to the orthogonality relations (1). The second one can be calculated with the aid of (3) by noting that $1/\bar{z} = z$ for $|z| = 1$. Thus, the above expression equals

$$\begin{aligned}
&-\frac{\phi_{n+1}(0)}{2\pi} \int_0^{2\pi} z^{n+1} \overline{\phi_{n+1}(z)}\, d\mu(t) \\
&= -\frac{\phi_{n+1}(0)}{2\pi} \overline{\int_0^{2\pi} \phi_{n+1}(z)\, \overline{z^{n+1}}\, d\mu(t)} \\
&= -\frac{\phi_{n+1}(0)}{2\pi} \overline{\int_0^{2\pi} \phi_{n+1}(z) \frac{1}{\kappa_{n+1}} \overline{\left(\kappa_{n+1} z^{n+1} + \sum_{l=0}^{n} \eta_{l,n+1} z^l\right)}\, d\mu(t)}
\end{aligned}$$

(cf. (31)); we were able to add the extra terms on the right-hand side since their integrals are zero in view of the orthogonality relations (1). Using the orthogonality relations again, the right-hand side here equals

$$-\frac{\phi_{n+1}(0)}{2\pi} \overline{\int_0^{2\pi} \phi_{n+1}(z) \frac{1}{\kappa_{n+1}} \overline{\phi_{n+1}(z)}\, d\mu(t)} = -\frac{\phi_{n+1}(0)}{\kappa_{n+1}} = -\Phi_{n+1}(0),$$

verifying (29). The proof is complete. ∎

A simple consequence of this lemma is

Corollary 11. *Let μ be an arbitrary finite positive Borel measure on the interval $[0, 2\pi)$ such that* $\operatorname{supp}(\mu)$ *is an infinite set. Then we have*

$$-\Phi_{n+1}(d\mu, 0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z^{1-n} \phi_n^2(d\mu, z)}{|\phi_{n+l}(d\mu, z)|^2}\, dt \qquad (z = e^{it}) \tag{33}$$

for $n = 0, 1, \ldots,$ and for every integer $l > 0$.

Proof. According to (19) the right-hand side of (33) equals to the right-hand side of (29). Therefore, the result follows from Lemma 10. ▮

We can now turn to the

Proof of Theorem 2. Let $f \in H^1$. The polynomial $\phi_n^*(\zeta)$ has no zeros for $|\zeta| < 1$; cf., e.g., [1, Theorem 2.1, p. 198] or [10, Theorem 11.4.1, p. 292]. Hence, writing $z = e^{it}$ as before, we have

$$\frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\,f(z)\,dt = \frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{\phi_n(\zeta)}{\phi_n^*(\zeta)}\,f(\zeta)\,d\zeta = 0. \tag{34}$$

Noting that $1/\bar{z} = z$ for $|z| = 1$, by (3) we have

$$|\phi_n(z)|^2 = \phi_n(z)\,\overline{\phi_n(z)} = z^{-n}\phi_n(z)\,\phi_n^*(z) \qquad (z = e^{it}),$$

that is,

$$z\,\frac{\phi_n(z)}{\phi_n^*(z)}\,|\phi_n(z)|^2 = z^{1-n}\phi_n^2(z) \qquad (z = e^{it}). \tag{35}$$

Therefore, by (34), the right-hand side of (4) can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\,\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{z^{1-n}\phi_n^2(z)}{|\phi_{n+l}(z)|^2}\,dt.$$

The right-hand side here is equal to $-\Phi_{n+1}(0)$ according to Corollary 11. The proof is complete. ▮

Theorems 2 (or 3) and 8 can be used to give a

Proof of Theorem 1. As $1/\bar{z} = z$ for $|z| = 1$, we have

$$|z\phi_n(z)/\phi_n^*(z)| = 1 \qquad \text{for} \quad |z| = 1; \tag{36}$$

we can conclude from the first inequality in (5) with $l = 1$ and $c = 1$ that

$$|\Phi_{n+1}(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} - 1\right|\,dt. \tag{37}$$

Thus (2) follows from (24). The proof is complete. ▮

6. Further Results About $\Phi_n(0)$ and Proof of Theorems 3 and 4

There are several consequences of Theorem 2 and of the closely related Lemma 10. The first result we are going to discuss was given as Theorem 1 by Rahmanov [8, p. 150]. We will write μ_s for the singular part of μ (that is, singular with respect to the restriction of the Lebesgue measure to Borel sets).

THEOREM 12. *Let μ be a finite positive Borel measure in the interval* $[0, 2\pi)$ *such that* $\operatorname{supp}(d\mu)$ *is an infinite set. Then we have*

$$|\Phi_{n+1}(0)| \leqslant \inf_{f\in H^1} \frac{1}{2\pi}\int_0^{2\pi} \big|\, |\phi_n(z)|^2 \mu'(t) - f(z)\big|\, dt + \frac{1}{2\pi}\int_0^{2\pi} |\phi_n(z)|^2\, d\mu_s(t) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \big|\, |\phi_n(z)|^2 \mu'(t) - 1\big|\, dt \qquad (z=e^{it}) \tag{38}$$

for $n = 0, 1, \ldots$.

This theorem together with (27) given another proof of Theorem 1.

Proof. Let $f \in H^1$. Writing $z = e^{it}$, by Lemma 10, (35), and (34) we obtain that

$$-\Phi_{n+1}(0) = \frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\, |\phi_n(z)|^2\, d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\, (|\phi_n(z)|^2 \mu'(t) - f(z))\, dt + \frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\, |\phi_n(z)|^2\, d\mu_s(t).$$

Taking the absolute values of the integrands, the first inequality in (38) follows in view of (36). As for the second inequality, observe that

$$\frac{1}{2\pi}\int_0^{2\pi} |\phi_n(z)|^2\, d\mu_s(t) = \frac{1}{2\pi}\int_0^{2\pi} |\phi_n(z)|^2\, d\mu(t) - \frac{1}{2\pi}\int_0^{2\pi} |\phi_n(z)|^2 \mu'(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} (1 - |\phi_n(z)|^2 \mu'(t))\, dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} \big|\, |\phi_n(z)|^2 \mu'(t) - 1\big|\, dt; \tag{39}$$

the second equality here follows from the orthogonality relations (1). Thus taking $f \equiv 1$ in the infimum in (38), the second inequality in (38) also follows. The proof is complete. ∎

Remark. We point out that the inequality

$$|\Phi_{n+1}(0)| \leqslant \text{const} \cdot \int_0^{2\pi} \big|\, |\phi_n(z)|^2 \mu'(t) - 1 \,\big| \, dt \qquad (z = e^{it})$$

which is somewhat weaker than (38) was also proved in [6, Corollary 1.4, p. 296] independently of [8].

We are now in a position to turn to the

Proof of Theorem 3. The first inequality follows from Theorem 2 and (36), while the second and fourth inequalities are obvious. We are going to prove the third inequality. To this end, let c be an arbitrary real number. Writing $z = e^{it}$ and using Schwarz's inequality, we obtain

$$\begin{aligned}
&\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} - c\right| dt\right)^2 \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}\left|\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} - c\,\frac{|\phi_{n+1}(z)|}{|\phi_n(z)|}\right| dt \\
&\quad \leqslant \frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2}\,dt \cdot \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|} - c\,\frac{|\phi_{n+1}(z)|}{|\phi_n(z)|}\right|^2 dt. \qquad (40)
\end{aligned}$$

The first integral on the right-hand side here is 1 according to (25). Expanding the square in the second integral and using (25) again, we can see that the right hand side here is

$$1 - 2c + \frac{c^2}{2\pi}\int_0^{2\pi}\frac{|\phi_{n+1}(z)|^2}{|\phi_n(z)|^2}\,dt. \qquad (41)$$

Express $\phi_n(z)/\phi_{n+1}(z)$ with the aid of the recurrence formula (32), and then take the squares of the absolute values of both sides. Noting that $|z| = 1$, and so we have $|\phi_n^*(z)/\phi_n(z)| = 1$ according to (36) (or (3) directly), we obtain

$$\frac{|\phi_{n+1}(z)|^2}{|\phi_n(z)|^2} = \frac{\kappa_{n+1}^2}{\kappa_n^2}\left(1 + 2\Re\left(z\,\overline{\frac{\phi_n^*(z)}{\phi_n(z)}}\,\overline{\Phi_{n+1}(0)} + |\Phi(0)|^2\right)\right).$$

Using (3) and the fact that $z = 1/\bar{z}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} z\,\overline{\frac{\phi_n^*(z)}{\phi_n(z)}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} z\,\frac{z^{-n}\phi_n(z)}{\overline{\phi_n(z)}}\,dt\,\frac{1}{2\pi}\int_0^{2\pi} z\,\frac{\phi_n(z)}{\phi_n^*(z)}\,dt = 0;$$

the last equality holds according to (34) with $f \equiv 1$. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_{n+1}(z)|^2}{|\phi_n(z)|^2}\,dt=\frac{\kappa_{n+1}^2}{\kappa_n^2}\,(1+|\Phi_{n+1}(0)|^2)=\frac{1+|\Phi_{n+1}(0)|^2}{1-|\Phi_{n+1}(0)|^2}. \tag{42}$$

To obtain the last equality here, note that

$$\kappa_{n+1}^2=\kappa_n^2+|\phi_{n+1}(0)|^2=\kappa_n^2+\kappa_n^2|\Phi_{n+1}(0)|^2;$$

see, e.g., [1, formula (V.1.17), p. 195] or [10, formula (11.3.6), p. 290]. Hence

$$\frac{\kappa_{n+1}^2}{\kappa_n^2}=\frac{1}{1-|\Phi_{n+1}(0)|^2}. \tag{43}$$

Thus, putting (40)–(42) together, we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2}-c\right|dt\leqslant\sqrt{1-2c+c^2\,\frac{1+|\Phi_{n+1}(0)|^2}{1-|\Phi_{n+1}(0)|^2}}$$

holds for every real number c. Finally, substituting

$$c=\frac{1-|\Phi_{n+1}(0)|^2}{1+|\Phi_{n+1}(0)|^2}>0$$

(cf. (43)) here, we obtain the third inequality in (5). The proof is complete. ∎

Proof of Theorem 4. The implication "$\Leftarrow$" in formula (6) follows immediately from Theorem 3. On the other hand, by the recurrence formula

$$\kappa_n\phi_{n+1}(\zeta)=z\kappa_{n+1}\phi_n(\zeta)+\phi_{n+1}(0)\,\phi_n^*(\zeta)$$

(cf. [10, formula (11.4.7), p. 293]) and by (43) the condition

$$\lim_{n\to\infty}\Phi_{n+1}(0)=0$$

implies

$$\lim_{n\to\infty}\frac{\phi_n(e^{it})}{\phi_{n+1}(e^{it})}=e^{-it}$$

uniformly for all real t. Hence "$\Rightarrow$" in (6) holds as well. The implication "$\Rightarrow$" in formula (7) is the same as Theorem 8 (cf. [4, Theorem 3, p. 64]). Finally, "$\Leftarrow$" in (7) was proved in [6, Theorem 1.1, p. 295]. The proof is complete. ∎

7. The Best Polynomial Approximation of Orthogonal Polynomials

The next theorem says that ϕ_n is close to what in a sense is the best polynomial approximation of ϕ_{n+l}.

THEOREM 13. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$ such that* $\mathrm{supp}(d\mu)$ *is an infinite set and let $l, n \geqslant 0$ be an integers. Given an arbitrary polynomial p of degree at most n, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(d\mu, z)|^2}{|\phi_{n+l}(d\mu, z)|^2}-1\right| dt$$
$$\leqslant 2\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|p(z)|}{|\phi_{n+l}(d\mu, z)|}-1\right)^2 dt} \qquad (z = e^{it}). \tag{44}$$

This result and its proof is motivated by Rahmanov's Lemma 2 in [8, p. 153].

Proof. Writing $z = e^{it}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}-1\right| dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}\left(\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-\frac{|p(z)|}{|\phi_n(z)|}\right)+\frac{|p(z)|}{|\phi_{n+l}(z)|}-1\right| dt$$
$$\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}\left|\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-\frac{|p(z)|}{|\phi_n(z)|}\right| dt$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|p(z)|}{|\phi_{n+l}(z)|}-1\right| dt. \tag{45}$$

Applying Schwarz's inequality to the first integral on the right-hand side we obtain

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}\left|\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-\frac{|p(z)|}{|\phi_n(z)|}\right| dt\right)^2$$
$$\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}\,dt\cdot\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+l}(z)|}-\frac{|p(z)|}{|\phi_n(z)|}\right)^2 dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|^2}{|\phi_{n+l}(z)|^2}\,dt-\frac{1}{\pi}\int_0^{2\pi}\frac{|p(z)|}{|\phi_{n+l}(z)|}\,dt+\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(z)|^2}{|\phi_n(z)|^2}\,dt. \tag{46}$$

The equality here follows by using (25) and then expanding the square in the second integral on the left-hand side. Noting that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(z)|^2}{|\phi_n(z)|^2}\,dt=\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(z)|^2}{|\phi_{n+l}(z)|^2}\,dt \quad (l\geqslant 0)$$

holds according to (19) and using (25) again, the right hand side of (46) can be seen to be equal to

$$1-\frac{1}{\pi}\int_0^{2\pi}\frac{|p(z)|}{|\phi_{n+l}(z)|}\,dt+\frac{1}{2\pi}\int_0^{2\pi}\frac{|p(z)|^2}{|\phi_{n+l}(z)|^2}\,dt=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{|p(z)|}{|\phi_{n+l}(z)|}-1\right|^2 dt.$$

To sum up, the right-hand side here is greater than or equal to the square of the first integral on the right-hand side of (45). A simple application of Schwarz's inequality shows that the right-hand side here is also greater than the square of the second integral on the right-hand side of (45). Hence (44) follows. The proof is complete. ▮

8. More About the Erdős Class

In this section, we give some more information about orthogonal polynomials associated with the Erdös class of measures, that is, those measures on $[0, 2\pi)$ for which $\mu'>0$ almost everywhere in this interval. As in Section 6, μ_s will denote the singular part of μ. Our first result is

THEOREM 14. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$, and assume that $\mu'>0$ almost everywhere in this interval. Let f be a 2π-periodic continuous function. Then, writing $z=e^{it}$, we have*

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{f(t)}{|\phi_n(d\mu, z)|}-1\right)^2 dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(f(t)\sqrt{\mu'(t)}-1\right)^2 dt+\frac{1}{2\pi}\int_0^{2\pi}f^2(t)\,d\mu_s(t). \tag{47}$$

Proof. The result follows simply by expanding the squares in the integrands on the both sides, and then using (20) above and the formula

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{f(t)}{|\phi_n(t)|}\,dt=\frac{1}{2\pi}\int_0^{2\pi}f(t)\sqrt{\mu'(t)}\,dt \qquad (z=e^{it}), \tag{48}$$

valid for every bounded measurable function f and every measure μ as described in the theorem to be proved. This formula is a direct consequence of the relation

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{1}{|\phi_n(z)|} - \sqrt{\mu'(t)} \right| dt = 0 \qquad (z = e^{it}),$$

given as formula (2.3) in [5, p. 243]. As shown in [5, p. 250], this formula can be easily obtained by Schwarz's inequality from (27) by using (20) with $f \equiv 1$. Note also that (48) is valid for an even wider class of functions (it is valid if $f \in L^2[0, 2\pi)$; cf. [5, formula (5.3), p. 251]). The proof is complete. ∎

The next result is Rahmanov's Lemmas 2 and 3 in [8, pp. 153–155].

THEOREM 15. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$. Then, writing $z = e^{it}$, for every polynomial p_n of degree at most n we have*

$$|\Phi_{n+1}(d\mu, 0)|^2 \leqslant \frac{2}{\pi}\int_0^{2\pi} \left(|p_n(z)| \sqrt{\mu'(t)} - 1 \right)^2 dt + \frac{2}{\pi}\int_0^{2\pi} |p_n(z)|^2 \, d\mu_s(t) \quad (49)$$

for $n = 0, 1, \ldots$.

Proof. If $\mu' > 0$ almost everywhere in $[0, 2\pi)$ then the result is a direct consequence of (5) (use the second inequality with $c = 1$), (44), and (47). Otherwise, apply (49) with $\mu_\varepsilon = \mu + \varepsilon m$ where m denotes the Lebesgue measure and $\varepsilon > 0$, and let ε tend to 0. The proof is complete. ∎

COROLLARY 16. *Let μ be a finite positive Borel measure in the interval $[0, 2\pi)$, and assume that $\mu' > 0$ almost everywhere in this interval. Then, writing $z = e^{it}$, we have*

$$|\Phi_{n+1}(d\mu, 0)|^2 \leqslant \min_{0 \leqslant j \leqslant n} \frac{4}{\pi}\int_0^{2\pi} \left(1 - |\phi_j(d\mu, z)| \sqrt{\mu'(t)} \right) dt \qquad (50)$$

for $n = 0, 1, \ldots$.

Proof. Using the preceding theorem with $p = \phi_j$ and expanding the square on the right-hand side of (49), we obtain

$$|\Phi_{n+1}(0)|^2 \leqslant \frac{2}{\pi}\int_0^{2\pi} |\phi_j(z)|^2 \mu'(t) \, dt$$
$$- \frac{4}{\pi}\int_0^{2\pi} |\phi_j(z)\sqrt{\mu'(t)} \, dt + \frac{2}{\pi}\int_0^{2\pi} dt + \frac{2}{\pi}\int_0^{2\pi} |\phi_j(z)|^2 \, d\mu_s(t).$$

Noting that the sum of the first and last terms on the right-hand side is equal to the third term in view of the orthogonality relations (1), formula (50) follows. The proof is complete. ▌

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REFERENCES

1. G. FREUD, "Orthogonal Polynomials," Pergamon, Oxford/New York/Toronto, 1971.
2. P. R. HALMOS, "Measure Theory," Graduate Texts in Mathematics, Vol. 18, Springer-Verlag, New York/Heidelberg/Berlin, 1974.
3. J. L. KELLEY, "General Topology," The University Series in Higher Mathematics, Van Nostrand, Toronto/New York/London, 1955.
4. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Asymptotics for the ratio of leading coefficients of orthonormal polynomials on the unit circle, *Constr. Approx.* **1** (1985), 63–69.
5. A. MÁTÉ, P. NEVAI, AND V. TOTIK, Strong and weak convergence of orthogonal polynomials, *Amer. J. Math.* **109** (1987), 239–281.
6. P. NEVAI, Characterization of measures associated with orthogonal polynomials on the unit circle, *Rocky Mountain J. Math.* **19** (1989), 293–302.
7. E. A. RAHMANOV, On the asymptotics of orthogonal polynomials, II, *Math. USSR Sb.* **46** (1983), 105–117; Russian original, *Mat. Sb.* **118** (1982), 104–117.
8. E. A. RAHMANOV, On the asymptotics of polynomials orthogonal on the unit circle with weights not satisfying Szegő's condition, *Math. USSR Sb.* **58** (1987), 149–167; Russian original, *Mat. Sb.* **130** (1986), 151–169.
9. W. RUDIN, "Real and Complex Analysis," 3rd. ed., McGraw-Hill Series in Higher Mathematics, McGraw–Hill, New York/London/Toronto, 1987.
10. G. SZEGŐ, "Orthogonal Polynomials," American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1967.